# An Unconditionally Stable Finite Element Time Domain Solution of Active Nonlinear Microwave Circuits Using Perfectly Matched Layers

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*Abstract* — **This paper proposes an extension of the unconditionally stable finite element time domain (FETD) method for the global electromagnetic analysis of active microwave circuits. This formulation has two advantages. First, the time step size is no longer governed by the spatial discretization of the mesh, but rather by the Nyquist sampling criterion. Second, the implementation of the truncation by the perfectly matched layers is straightforward. A benchmark test on a microwave amplifier indicates that this extended FETD algorithm is not only superior than FDTD-based algorithm in mesh flexibility and simulation accuracy, but also reduces computation time dramatically.**

## I. INTRODUCTION

Successful circuit design at microwave and millimeter wave frequencies requires considering electromagnetic coupling effects. This requirement can be fulfilled using a full-wave approach, which solves Maxwell's equations while taking into account the interaction between electromagnetic waves and lumped elements comprehensively. Much effort has been devoted to the extension of the FDTD method to incorporate lumped microwave devices into 3-D full wave analysis [1]. Recently, the FETD algorithm [2] has been successfully applied in combination with SPICE to solve microwave circuits including lumped elements. Chang [3] used FETD to derive equivalent current sources and capacitances of distributed circuits to be combined with state equations of lumped element active microwave devices. Although this scheme gives better accuracy than FDTD-based techniques reported so far, the time step required to converge to a final solution is smaller than those required by explicit FDTD methods due to conditionally stability [4]. In this paper, an unconditionally stable FETD solution of microwave passive/active circuit based on Gedney's method [5] is presented. Combining the anisotropic PML approach in the frequency domain as presented by Z. Sack *et al* [1] and in the time domain as in Mathis' method [7], we also present how an anisotropic PML can be implemented in this unconditionally stable FETD scheme.

## II. THEORY

### A. Anisotropic PML for FETD Formulation

This section will derive the finite element time domain formulation to be used when the perfectly matched anisotropic absorber is employed to terminate the computational domain. The derivation here is based on [6] and [7]. The general time-harmonic form of Maxwell's equations is

$$\begin{aligned}\nabla\times\mathbf{E} &= -j\omega[\mu]\mathbf{H}-[\sigma_M]\mathbf{H} = -j\omega\left[\bar{\mu}\right]\mathbf{H}\\ \nabla\times\mathbf{H} &= j\omega[\varepsilon]\mathbf{E}+[\sigma_E]\mathbf{E}+\mathbf{J}_i = j\omega\left[\bar{\varepsilon}\right]\mathbf{E}+\mathbf{J}_i\end{aligned} \tag{1}$$

where $\left[\bar{\mu}\right]$ and $\left[\bar{\varepsilon}\right]$ are complex diagonal tensors of permeability and permittivity, respectively. In this section, we concentrate on the materials with $\left[\bar{\mu}\right]$ and $\left[\bar{\varepsilon}\right]$ diagonal in the same coordinate system. To match the intrinsic impedance of the anisotropic PML medium to free space, the condition:

$$\frac{\left[\bar{\varepsilon}\right]}{\varepsilon_0} = \frac{\left[\bar{\mu}\right]}{\mu_0} \tag{2}$$

must hold. Consequently, the tensors $\left[\bar{\varepsilon}\right]$ and $\left[\bar{\mu}\right]$ can be written as

$$\left[\bar{\varepsilon}\right] = \varepsilon[\Lambda]\ ,\ \left[\bar{\mu}\right] = \mu[\Lambda] \tag{3}$$

where

$$[\Lambda] = \begin{bmatrix} S_yS_z/S_x & 0 & 0\\ 0 & S_xS_z/S_y & 0\\ 0 & 0 & S_yS_x/S_z\end{bmatrix},\quad S_{x,y,z} = 1+\frac{\sigma_{x,y,z}}{j\omega\varepsilon_0}$$

Since we want to consider only the electric field in the calculation domain, Maxwell's equations can be written as

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$$\nabla\times\frac{1}{\mu}\left([\Lambda]^{-1}\cdot\nabla\times\mathbf{E}\right)-\omega^2\varepsilon[\Lambda]\mathbf{E}=-j\omega\mathbf{J_i} \quad (4)$$

Using (4) as the starting point, we can obtain a modified equation which can be utilized be derive the time domain formulation. Next, we want to isolate the dependences in ω in order to go to a temporal formulation

$$\nabla\times\frac{1}{\mu}\nabla\times\mathbf{E}+\frac{2}{j\omega\varepsilon_0}\nabla\times\frac{1}{\mu}[I]\nabla\times\mathbf{E}-\frac{1}{\omega^2\varepsilon_0^2}\nabla\times\frac{1}{\mu}[I]^2\nabla\times\mathbf{E}=\omega^2\varepsilon\mathbf{E}+$$
$$\frac{2\omega\varepsilon}{j\varepsilon_0}[J]\mathbf{E}-\frac{\varepsilon}{\varepsilon_0^2}\left([J]+2[K]\right)\mathbf{E}-\frac{2\varepsilon}{j\omega\varepsilon_0^2}[L]\mathbf{E}+\frac{\varepsilon}{\omega^2\varepsilon_0^4}[K]^2\mathbf{E}-j\omega\mathbf{J_i} \quad (5)$$

where

$$[I]=\begin{bmatrix}\sigma_x & 0 & 0\\ 0 & \sigma_y & 0\\ 0 & 0 & \sigma_z\end{bmatrix},\ [J]=\begin{bmatrix}\sigma_y+\sigma_z & 0 & 0\\ 0 & \sigma_z+\sigma_x & 0\\ 0 & 0 & \sigma_x+\sigma_y\end{bmatrix}$$
$$[K]=\begin{bmatrix}\sigma_y\sigma_z & 0 & 0\\ 0 & \sigma_z\sigma_x & 0\\ 0 & 0 & \sigma_x\sigma_y\end{bmatrix}, \quad (6)$$
$$[L]=\begin{bmatrix}(\sigma_y+\sigma_z)\sigma_y\sigma_z & 0 & 0\\ 0 & (\sigma_z+\sigma_x)\sigma_z\sigma_x & 0\\ 0 & 0 & (\sigma_x+\sigma_y)\sigma_x\sigma_y\end{bmatrix}$$

The next step is to convert the frequency domain formulation to the time domain version using the following relations:

$$j\omega\leftrightarrow\frac{\partial}{\partial t},-\omega^2\leftrightarrow\frac{\partial^2}{\partial t^2} \quad (7)$$
$$\frac{1}{j\omega}\leftrightarrow\int_0^t,\frac{-1}{\omega^2}\leftrightarrow\int_0^t$$

Applying the above transformation, we can recast the Maxwell's equation in the following form

$$\nabla\times\frac{1}{\mu}\nabla\times\mathbf{E}+\frac{2}{\varepsilon_0}\nabla\times\frac{1}{\mu}[I]\int_t\nabla\times\mathbf{E}+\frac{1}{\omega^2\varepsilon_0^2}\nabla\times\frac{1}{\mu}[I]\iint_t\nabla\times\mathbf{E}=\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}$$
$$-\frac{2\varepsilon}{\varepsilon_0}[J]\frac{\partial\mathbf{E}}{\partial t}-\frac{\varepsilon}{\varepsilon_0^2}([J]+2[K])\mathbf{E}-\frac{2\varepsilon}{\varepsilon_0^3}[L]\int_t\mathbf{E}-\frac{\varepsilon}{\varepsilon_0^4}[K]^2\iint_t\mathbf{E}-\frac{\partial\mathbf{J}_i}{\partial t} \quad (8)$$

Testing equation (8) with edge basis function $\mathbf{W}^{(1)}$ associated with non-PEC edges of the grid, integrating by parts, and decomposing the unknown field **E** in a finite element basis yields the following weak form

$$[A]\underline{e}+[B]\frac{d\underline{e}}{dt}+[C]\frac{d^2\underline{e}}{dt^2}+[D]\underline{f}+[E]\underline{g}=-I \quad (9)$$

where

$$(A)_{ij}=\int_V\frac{1}{\mu}\nabla\times W_i^{(1)}\cdot\nabla\times W_j^{(1)}+\frac{\varepsilon}{\varepsilon_0^2}\left([J]^2+2[K]\right)W_i^{(1)}\cdot W_j^{(1)}dV$$
$$(B)_{ij}=\int_V\frac{2\varepsilon}{\varepsilon_0}[J]W_i^{(1)}\cdot W_j^{(1)}dV$$
$$(C)_{ij}=\int_V W_i^{(1)}\cdot W_j^{(1)}dV$$
$$(D)_{ij}=\int_V\frac{2}{\mu\varepsilon_0}[I]\nabla\times W_i^{(1)}\cdot\nabla\times W_j^{(1)}+\frac{2\varepsilon}{\varepsilon_0^3}[L]W_i^{(1)}\cdot W_j^{(1)}dV$$
$$(E)_{ij}=\int_v\frac{1}{\mu\varepsilon_0^2}[I]^2\nabla\times W_i^{(1)}\cdot\nabla\times W_j^{(1)}+\frac{\varepsilon}{\varepsilon_0^4}[K]^2W_i^{(1)}\cdot W_j^{(1)}dV$$
$$(I)_i=\int_V W_i^{(1)}\cdot\frac{\partial\mathbf{J}_i}{\partial t}dV$$

which are all time-independent matrices.

$$\underline{f}=\int_t\underline{e}\quad,\quad\underline{g}=\iint_t\underline{e}$$

are the coordinates of the unknown field in the finite element basis.

In the results presented by Gedney[8], it was clearly pointed out that the mismatch at the PML interface can be significantly reduced by carefully selecting the conductivity, σ, of the material.

### *B. State Equation-FETD Combination*

In order to introduce lumped elements into the 3D FETD simulator, all vectors and matrices have been split to separate the unknowns associated to edges in the mesh in which the lumped elements are located (denoted by the subscript *c*), from those associated to standard edges (subscript *e*). Based on the method mentioned in [3], (9) can be recast into the following form

$$\begin{bmatrix}[A]_{ee}\,[A]_{ec}\\ [A]_{ce}\,[A]_{cc}\end{bmatrix}\begin{bmatrix}\underline{e_e}\\ \underline{e_c}\end{bmatrix}+\begin{bmatrix}[B]_{ee}\,[B]_{ec}\\ [B]_{ce}\,[B]_{cc}\end{bmatrix}\cdot\frac{\partial}{\partial t}\begin{bmatrix}\underline{e_e}\\ \underline{e_c}\end{bmatrix}+\begin{bmatrix}[C]_{ee}\,[C]_{ec}\\ [C]_{ce}\,[C]_{cc}\end{bmatrix}\cdot \quad (10)$$
$$\frac{\partial^2}{\partial t^2}\begin{bmatrix}\underline{e_e}\\ \underline{e_c}\end{bmatrix}+\begin{bmatrix}[D]_{ee}\,[D]_{ec}\\ [D]_{ce}\,[D]_{cc}\end{bmatrix}\begin{bmatrix}\underline{f_e}\\ \underline{f_c}\end{bmatrix}+\begin{bmatrix}[E]_{ee}\,[E]_{ec}\\ [E]_{ce}\,[E]_{cc}\end{bmatrix}\begin{bmatrix}\underline{g_e}\\ \underline{g_c}\end{bmatrix}=\begin{bmatrix}\underline{I_e}/\partial t\\ \underline{I_c}/\partial t\end{bmatrix}$$

where $[B]_{ec},[B]_{ce},[B]_{cc},[D]_{ec},[D]_{ce},[D]_{cc},[E]_{ec},[E]_{ce},[E]_{cc}$, $\underline{f_c}$ and $\underline{g_c}$ are zero since the lump circuits are not inside the perfectly matched anisotropic absorber. Based on the Newmark-Beta formulation [9][10], (10) is approximated as

$$[A]_{ee}[\beta\underline{e}_e^{n+1}+(1-2\beta)\underline{e}_e^n+\beta\underline{e}_e^{n-1}]+[A]_{ec}\underline{e}_c+[B]_{ee}\frac{\underline{e}_e^{n+1}-\underline{e}_e^{n-1}}{2\Delta t}+ \quad (11)$$
$$[C]_{ee}\frac{\underline{e}_e^{n+1}-2\underline{e}_e^n+\underline{e}_e^{n-1}}{\Delta t^2}+[C]_{ec}\frac{\partial^2\underline{e}_c}{\partial t^2}+[D]_{ee}f_e^n+[E]_{ee}g_e^n=-\frac{\partial}{\partial t}\underline{I}_e$$

$$[A]_{ce}[\beta \underline{e}_e^{n+1} + (1-2\beta)\underline{e}_e^{n} + \beta \underline{e}_e^{n-1}] + [A]_{cc}\,\underline{e}_c + [C]_{ce}\frac{\underline{e}_e^{n+1} - 2\underline{e}_e^{n} + \underline{e}_e^{n-1}}{\Delta t^2} + [C]_{cc}\frac{\partial^2 \underline{e}_c}{\partial t^2} = -\frac{\partial \underline{I}_e}{\partial t} \quad (12)$$

where $f_e^n = f_e^{n-1} + \frac{\Delta t}{2}\underline{e}_e^{n-1} + \frac{\Delta t}{2}\underline{e}_e^{n}$,

$$g_e^n = g_e^{n-1} + \Delta t \underline{e}_e^{n-1} + \frac{\Delta t^2}{4}\underline{e}_e^{n-1} + \frac{\Delta t^2}{4}\underline{e}_e^{n} \quad (13)$$

Gedney[5] has proved that unconditional stability is achievable, providing the interpolation parameter $\beta \geq 1/4$. it was further shown that choosing $\beta = 1/4$ minimized solution error. Substituting $\underline{e}_e^{n+1}$ in (11) into (12) yields a set of 2nd order ordinary differential equations in terms of unknown $\underline{e}_c$.

$$[C]\frac{\partial^2 \underline{e}_c}{\partial t^2} + \frac{\partial \underline{I}_c}{\partial t} = [F] \quad (14)$$

To solve the above 2nd order ODE, let

$$\underline{I}_d = [C]\frac{\partial \underline{e}_c}{\partial t} + \underline{I}_c \quad \text{and} \quad \frac{\partial \underline{I}_d}{\partial t} = [F] \quad (15)$$

which can be interpreted as a generalized Kirchoff law involving electric field and current at the terminals of the lumped elements included in the computational domain. (15) can be combined with the state equations of active/passive linear/nonlinear lumped element components to determine $\underline{e}_e$ and $\underline{I}_d$.

The computation procedure of the mixed electromagnetic and circuit simulation algorithm can be summarized as follows. Assume all the electric field quantities are known at time steps n-1 and n, and they are used to set up the known forcing vector $[F]$ in equation (14). Currents, $\underline{I}_d$, and voltages, $\underline{e}_e$, across the lumped elements at time step $n$+1 are obtained by simultaneous solution of equations (15). The nonlinear differential equation (15) is discretized with a finite difference scheme, and then the resulting nonlinear system of equations is solved using a modified Powell hybrid algorithm and a finite-diffenrence approxiation to the Jacobian, provided by an IMSL routine. Once the combined system of equations (15) is solved, then currents and voltages across lumped elements are computed. They are fed back into equation (11) to compute the electric field at every edge in the grid at time step $n$+1. This time stepping scheme is repeated until the observation point after the drain of the device records the complete output response.

## III. RESULT

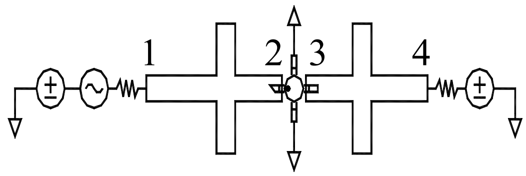


Fig. 1. The structure of a microwave amplifier.

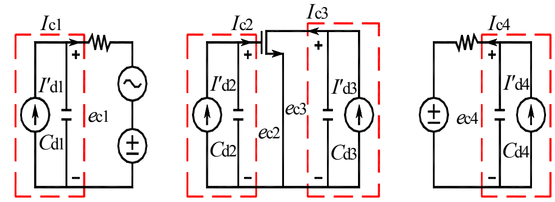


Fig. 2. The equivalent circuit ($Nc$ = 4).

A microwave amplifier has been analyzed using this technique. Fig. 1 shows the layout of the simulated circuit using this technique. For comparison purposes, the circuit and the large signal model of the MESFET used are the same as that employed with the FETD technique in [3] the extended FDTD technique in [11]. The circuit is shielded and the end wall is terminated by a twenty-cell PML medium. $\sigma$ is chosen to be specially variant along the normal axis as

$$\sigma = \frac{4\,|z - z_0|^2}{d^2}$$

where $z_0$ is the interface, d is the depth of the PML.

The DC bias condition of this amplifier, $V_{gs}$ = -0.81V and $V_{ds}$ = 6.4 V, is provided by the two power supplies connected at gate and drain, while the RF source is represented by the generator connected to the gate. A modulated Gaussian pulse with central frequency at 5 GHz is used to excite the circuit. The mixed electromagnetic and circuit simulation algorithm outlined in the previous sections requires considering electromagnetic/circuit interaction at four locations: the input generators, the device input (gate) and output (drain) terminals, and the load/drain power supply. Fig. 2 shows the three equivalent circuits used. The first models interactions between the input generator and the input microstrip line; the second describes interactions between the MESFET and its input and output microstrip lines, and the third account for the coupling between the load and drain power supply with the output microstrip line. The distributed passive part of the microwave amplifier is represented via the Norton generators with capacitive internal admittance. The value of the Norton sources and their internal capacitance is derived by the matrix equation (14).

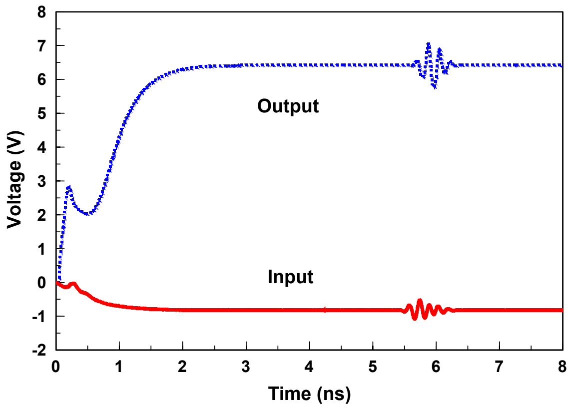

Fig. 3. Time response of the microwave amplifier

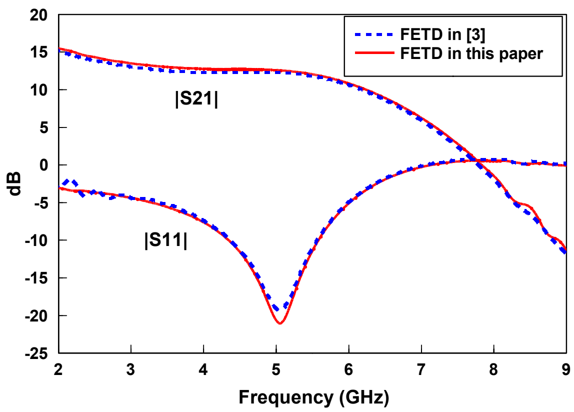

Fig. 4. Small-signal analysis of the microwave amplifier.

A FORTRAN program has been implemented. The number of the unknown electric fields to be solved in this mesh is 303198. The mesh had a minimum tetrahedral radius of h=25 mil. The time step size, 3.5 ps, is 10 times bigger than that in [4] and the number of computation time steps is reduced to 2500. The running time is 50 minutes on a PC Pentium II 450 MHz. Fig. 3 shows the time domain response of the amplifier when a Gaussian modulated pulse is applied at its input. Fig. 4 shows a good comparison of the scattering parameters of the amplifier with the result obtained with the FETD method in [4] shown.

## IV. CONCLUSION

The extended FETD algorithm based on an unconditionally stable solution of the vector wave equation has been used to analyze a microwave amplifier in this paper. The result not only retains the accuracy of the FETD method in [4] but also shows that the number of time iterations required can be significantly reduced. As long as the optimal parameters of perfectly matched layer are found, this implicit method can be competitive with an explicit method.

## REFERENCES


[1] P. Ciampolini, P. Mezzanotte, L. Roselli, R. Sorrentino "Accurate and efficient circuit simulation with lumped-element FDTD technique," *IEEE Transactions on Microwave Theory Tech.*, vol. 44, (no.12, pt.1), pp. 2207-2215, Dec. 1996.

[2] K. Guillouard, M. F. Wong, and V. Fouad Hanna, "A New Global Time Domain Electromagnetic Simulator of Microwave Circuits Including Lumped Elements Based on Finite Element Method," *1997 IEEE MTT-S Digest,* pp. 1239-1242.

[3] S.-H. Chang, R. Coccioli, Y. Qian, and T. Itoh, "A Global Finite-Element Time Domain Analysis of Active Nonlinear Microwave," *IEEE Trans. Microwave Theory Tech.*, vol.47, (no.12), pp.2410-16, Dec. 1999.

[4] S. D. Gedney and U. Navsariwala, "A comparison of the performance of the FDTD, FETD, and planar generalized Yee algorithms on high performance parallel computers," Int. J. Numerical Modeling (Electronic Networks, Devices and Fields), vol. 8, pp. 265-276, May-Aug, 1995.

[5] A. D. Gedney, and U. Navsariwala, "An Unconditionally Stable Finite Element Time-Domain Solution of the Vector Wave Equation," *IEEE Microwave and Guided Wave Letters*, vol.5, (no.10), pp.332-334, Oct. 1995.

[6] Z. S. Sacks, D.M. Kingsland, R. Lee, and Jin-Fa Lee, "A perfectly matched anisotropic absorber for use as an absorbing boundary condition," *IEEE Trans. Antennas Propagat.*, vol.43, (no.12), IEEE, pp.1460-1463, Dec. 1995.

[7] V. Mathis, "An Anisotropic Perfectly Matched Layer-Absorbing Medium In Finite Element Time Domian Method For Maxwell's Equations," *IEEE Antennas and Propagat. Society International Symposium. Digest,* pp.680-3 vol.2, 1997

[8] S. D. Gedney, " An Anisotropic Perfectly Matched Layer-Absorbing Medium for the Truncation of FDTD Lattics," ," *IEEE Trans. Antennas Propagat.,* vol.44, (no.12), pp.1630-1639, Dec. 1996.

[9] N. M. Newmark, "A method of computation for structural dynamics." *J. Engineering Mechanics Division, ASCE*, vol. 85, pp. 67-94, July 1959.

[10] O. C. Zienkiewicz, "A new look at the Newmark, Houboult and other time stepping formulas. A weighted residual approach," *Earthquake Engineering and Structural Dynamics*, vol. 5, pp. 413-418, 1977.

[11] C.-N. Kuo, B. Houshmand, and T. Itoh, "Full-Wave Analysis of Packaged Microwave Circuits with Active and Nonlinear Devices: An FDTD Approach," *IEEE Trans. Microwave Theory Tech.*, vol. 45, pp. 819-826, May 1997.